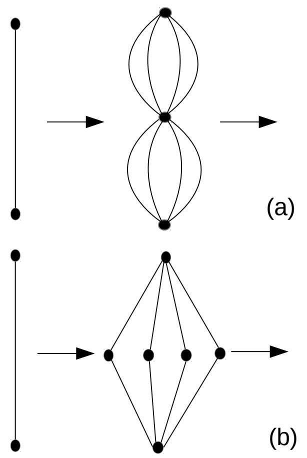
Spin models on hierarchical lattices



Migdal-Kadanoff real space renormalization group (RG) proceudre becomes exact on the so-called hierarchical lattices. The purpose of this project is to consider the critical properties of the Ising model defined on these lattices using the RG recursion relation we have learned in the lecture. Two examples of the hierarchical lattice are depicted in the Figure. They are constructed in a recursive way. Starting from a single bond, one replaces this by p strings of b links at the first step of iteration ($n = 1$). For the two lattices in Figure (a) and (b), $p = 4$ and $b = 2$. This proceudre is iterated infintely.

If we measure the length by the steps on the lattice, the length increases by the factor of b at each iteration. By counting the number of bonds N and the length L at the n-th iteration and using the relation $N = L^d$, find the dimension d of these lattices.

You can consider many more examples of the hierarchical lattice in, for example, [M. Kaufman and R. B. Griffiths, Phys. Rev. B **24**, 496 (1981)]. Try to find the dimension of these lattices.

Now try to construct the Hamiltonian for the Ising spins ($\sigma_i = \pm 1$) on a hierarchical lattice interacting via the nearest-neighbor interaction. Find and study the RG recursion relation and the RG flow for the coupling constants and obtain the critical exponents. Study their dependence on the dimensionality of the hierarchical lattice. What happens when there is an external field?

What about other fractal lattices, e.g. Sierpinsky carpet? Can you devise a way to obtain a closed form of RG recursion relation for the Ising model defined on these lattices?